Analytical Velocity Profile in Tube for Laminar and Turbulent Flow

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Abstract A new analytical formula of the velocity profile for the laminar and turbulent flow in a tube with a circular cross-section will be introduced in this article. This formula is rather simple and it can be improved. This new formula will also be compared with two different power law formulas. The advantage of this new formula is that it can also be compared with the log lawnear the wall.

1 Introduction

The author is dealing with a fluid flow in a straight tube with a circular cross-section governed by the pressure gradient in this paper. Some fundamental ideas of the new velocity profile derivation and its comparison with current analytical formulas of the turbulent velocity profile will also be presented and discussed here.

First the formula for the laminar velocity profile has to be mentioned. It is possible to find this derivation in each book dedicated to the fluid mechanics. The laminar velocity profile of the fluid flow governed by the pressure gradient is parabolic.

$$v = 2.v_{(av)}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{1}$$

Where R is the tube radius, $v_{(av)}$ is the average velocity over the cross-section. It is assumed that there is only one velocity component in the tube axis direction.

Many researches have been trying to find an analytical solution of turbulent velocity profile in the tube. They are known as power law velocity profiles. The formulas used in [3] and [2] will be used here as the representatives of all these formulas. First formula is

$$v = \frac{v_{(av)}}{2}.\left(\frac{1}{n}+1\right).\left(\frac{1}{n}+2\right)\left[1-\frac{r}{R}\right]^{1/n} \tag{2}$$

Where n is a coefficient which is a function of the Reynolds number. It has to be determined on the basis of the experimental data. The value $n=7$ is reasonable for many practical flow approximations. The expression before the square bracket is the maximum velocity in the tube center.

The next analytical formula is

$$v = v_{(av)}.(n_0+1).\left[1-\left(\frac{r}{R}\right)^2\right]^{n_0} \tag{3}$$

The expression before square the bracket is also the maximum velocity in the tube center. The coefficient n_0 is also a function of the Reynolds number and it can be expressed by the formula:

$$\frac{1}{n_0} = 1+\sqrt[6]{\frac{Re}{50}} \tag{4}$$

These two analytical formulas for turbulent velocity profile in a straight tube have some fundamental discrepancies. Both of them have a problem near the tube wall. This problem is in the infinite derivative valueon the wall which means that there is infinite shear stress on the wall. Which mean infinite friction losses in the tube.

Next discrepancy is related to the velocity profile smoothness at the tube center. This is the problem of the formula used by Munson. The first derivative of this formula is not smooth in the tube center. The utilizing of these formulas is restricted because of these fundamental discrepancies.

The problem near the wall has to be solved in a different way using other empirical formulas which are valid only near the wall. There is a boundary layer near the wall. The boundary layer is an area of the flow near the wall where the friction (viscous) forces are proportional to the dynamic forces. It can be also said that it is an area near the wall where the vorticity is not zero. It means that $\mathrm{curl}\ \boldsymbol{v}$ is not zero. This boundary layer can be divided into three areas. The first of them is the one which is nearest to the wall.It is called viscous sub-layer. The second one is the transition area and the third one is the turbulent boundary layer. It is necessary to define some quantities in order to be able to describe the velocity profile in the boundary layer.

$$v^* = \sqrt{\frac{\tau_w}{\nu}} \tag{5}$$

Where v^* is the shear velocity, τ_w is shear stress on the wall, ν is cinematic viscosity. Then it is possible to define the dimensionless distance from the wall (y^+) and the dimensionless velocity (v^+).

$$v^+ = \frac{v}{v^*} \tag{6}$$

$$y^+ = \frac{v^*.y}{\nu} \tag{7}$$

Where y is the distance from the wall.

The dimensionless velocity profile in viscous sublayer can be then expressed this way

$$v^+ = y^+ \tag{8}$$

The dimensionless velocity profile in the turbulent boundary layer can be expressed by the log law.

$$v^+ = \frac{1}{\kappa}\log(y^+) + B \tag{9}$$

Different authors are using different values of the constants κ and B. For example the coefficients used:$\kappa=0,4$ and $B=5$ in the book [3], $\kappa= 0,173$ and $B = 5,5$ in the book [2], $\kappa=0,41$ and $B=5,2$ in the book [1].

2 Introduction of a new velocity profile

The new velocity profile is based on the vorticity density $\vec{\gamma}$ distribution over the tube cross-section. This vorticity induces the velocity. The relationship between the induced velocity and the vorticity density is through the Biot-Sawart law. The velocity induced by the circular vortex filaments with constant vorticity γ_i aligned in vortex tube with infinite length has to be expressed first. The situation is depicted in the **fig. 1**.

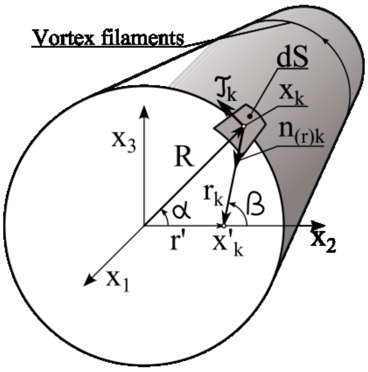

Fig. 1 Circular vortex filaments aligned in the infinite length vortex tube.

The velocity induced by a vorticity closed inside the infinitesimal area dS of the vortex tube with radius R can be expressed.

$$v_i = \int_S \frac{\varepsilon_{ijk}.\gamma_j.(x'_k - x_K).dS}{2.\pi.r^3} \tag{10}$$

The Einstein summation convection is applied to the previous Biot-Sawart law expression. The meaning of the quantities in the expression (10) is as follows; γ_i is the vorticity density vector, x'_k are location coordinates of the induced velocity, x_k are location coordinates of the vorticity density vector, dS is an infinitesimal area with constant vorticity density vector, r is the distance between point x'_k and x_k. Area of integration S is an infinite cylindrical surface.

It is possible to find the analytical solution of this integral. The velocity induced by this vortex tube can be then expressed.

$$v_i = 2.\gamma.\varepsilon_{ijk}.\tau_j.n_{(r)k} \tag{11}$$

Where the τ_j is a unit vector tangential to the vortex filament, $n_{(r)k}$ is the unit vector in r direction. The solution of a vector product $\varepsilon_{ijk}.\tau_j.n_{(r)k}$is the unit vector in the tube axis direction.

Now it is necessary to express the velocity for the case when the vorticity density $\gamma$is a function of radius inside of the tube. The variable radius will now be marked as r (radius of the vortex tube). The radius of the tube will be marked R. It means that there will be a continuous distribution of the vorticity density over the cross-section. It will be assumed, that the vorticity density distribution will be a polynomial function of variable r.

$$\gamma = \sum_{n=0}^{N} A_{(n)}.r^n \tag{12}$$

The coefficients $A_{(n)}$ will be determined from the boundary conditions(slip condition on the wall), from a given flow rate through tube, and from the condition of smoothness. The velocity profile in a tube can then be expressed.

$$v = v_{(av)}.\frac{(N+3)}{(N+1)}.\left[1 - \left(\frac{r}{R}\right)^{N+1}\right] \tag{13}$$

It is possible to compare this expression with the (1), (2) and (3). When *N = 1* then it is the expression for a laminar velocity profile, for *N > 1* it is turbulent velocity profile and for *N*→∞ it is a piston profile, it is the case of the infinite Reynolds number. *N* can be expressed from the pressure drop and flow rate in the tube.

$$N = \frac{R^2}{2.\mu.v_{(av)}}.\frac{p_1 - p_2}{L} - 3 \tag{14}$$

3 Discusion

It is possible to compare velocity profiles (2), (3) and (13), see **Fig 2**. The comparison is done for Re = 10,186. The value of the *n = 6*, for this Re, is taken into consideration in case of the Munson's power law velocity profile. The velocity profiles are normalized by the average velocity $v_{(av)}$.

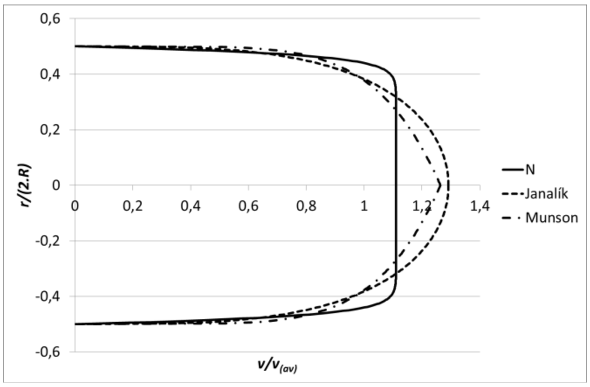

Fig. 2 The comparison of velocity profiles

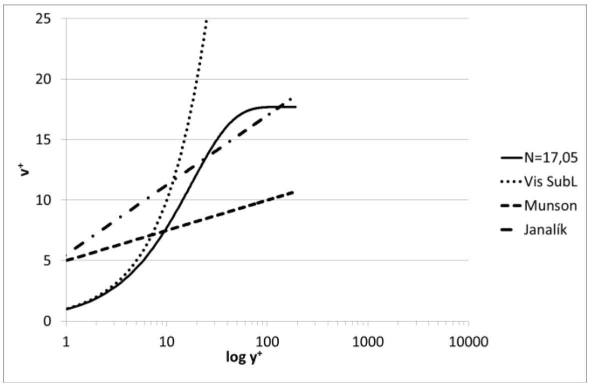

Fig.3 The Comparison of velocity profile near the wall with the log law.

It is apparent that all three velocity profiles are different. The new velocity profile hasa problem in the tube center because the second derivative is zero there. It means that the radius of curvature is also zero in that location. This is not realistic, but there is achance to remove this discrepancy and the author is working on this problem.

It is also possible to comparethe new velocity profile with log law and viscous sub-layer in area near the wall. This comparison is outlined in the **Fig. 3**. It is not possible to do this comparison with the power law velocity profiles because they have the infinite derivative near the wall. It is apparent that the comparison with the log law is not bad. When the new velocity profile is improved, as mentioned in the previous paragraph, then the agreement with the log law will also be improved because the curve of the new velocity profile will be more flat.

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